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ON SHOCK WAVE PROPAGATION IN STRESSED ISOTROPIC NONLINEARLY ELASTIC MEDIA*

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The relationships in weak shock waves propagating over an arbitrary elastic medium in lightly deformed (stressed) state are analyzed in nonlinear approximation. Hugoniot curves that correspond to waves propagating over an arbitrary stressed state are investigated using relations at discontinuities in quasilongitudinal and quasitransverse shock waves. Generally, when the initial deformation ahead of the shock wave disrupts the isotropy in planes parallel to the wave front, the Hugoniot curve represents for both the quasitransverse and quasilongitudinal waves a certain curve which is investigated below. Points that correspond to shock waves accompanied by entropy rise are indicated on segments of that curve. In the case of quasitransverse waves this made necessary and taking into account in computations fourth order terms with respect to discontinuity amplitude. The shock wave velocity behavior and its relation to small perturbation velocities is investigated. Segments of the Hugoniot curve are indicated at whose points the conditions of /shock wave/ evolution are satisfied. It is shown that in the case of quasitransverse waves the conditions of evolution and those of entropy increase correspond to two different sets on the Hugoniot curve. Only shock waves that correspond to the intersection of these sets can actually exist.

Shock waves in nonlinearly elastic media were investigated in /1—5/. (The numerous publications in which purely longitudinal shock waves, i.e. waves in which only the normal component becomes discontinuous and those in which discontinuities were analyzed in linear approximation, are not mentioned here.) Weak shock waves propagating in an isotropic elastic medium in unstressed state were the subject of detailed analysis in /1/. Some of the results obtained in /2—5/ are relevant to the present investigation. The dependence of shock wave velocity on the discontinuity of the derivative of the displacement normal vector component along the normal to the wave, uniquely related to density, was investigated, and the change of the entropy sign for quasilongitudinal waves was clarified. When the initial deformation is small relative to the discontinuity amplitude, the third approximation does not indicate a change of entropy in quasilongitudinal waves. The equation of momentum conservation and the defining equations of the medium showed the existence of particular types of shock waves: pure longitudinal and pure transverse. Certain corollaries were obtained in the form of inequalities. The Hugoniot curve was not considered as a whole in /2—5/, and shock wave evolution was not investigated.

1. Conditions on shock waves.

Properties of an elastic medium are fully determined by its elastic potential, i.e. by function $\Phi = \rho_0 U(\varepsilon_{ij}, S)$, where U and S are, respectively, the internal energy and entropy of a unit of mass, ρ_0 is the density in the undeformed state, i.e. when $\varepsilon_{ij} = 0$, and ε_{ij} is the tensor of finite strains which can be defined as follows:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial \xi^j} + \frac{\partial w_j}{\partial \xi^i} + \frac{\partial w^s}{\partial \xi^i}\frac{\partial w_s}{\partial \xi^j}\right)$$

where $\xi^i = \xi_i$ are coordinates in some stationary orthogonal Cartesian coordinate system and $w^i = w_i$ are components of the vecotr of displacements of points of the body in that coordinate system, and considered to be functions of Lagrangian coordinates ξ^i and time t. Recurrent indices always imply summation.

In an isotropic body U depends on ε_{ij} only in terms of invariants of the tensor of ε_{ij}.

We define stress in the body in terms of the Piola—Kirchhoff asymmetric stress tensor, which enables us to determine the elasticity force acting on the surface in the current stress state by integrating with respect to ξ^i over the initially undeformed surface

$$F_i = \int_{\Sigma_0} \sigma_{ij} n_j \, d\Sigma_0$$

where n_j are components of the vector normal to $d\Sigma_0$ in the orthogonal Cartesian system of coordinates ξ_i.

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It was shown in /6/ that

$$\sigma_{ij} = \frac{\partial\Phi}{\partial(\partial w_i/\partial\xi_j)} = \frac{\partial\Phi}{\partial\varepsilon_{mn}}\frac{\partial\varepsilon_{mn}}{\partial(\partial w_i/\partial\xi_j)} \tag{1.1}$$

When analyzing the relations on discontinuities propagating over the initially undeformed state, it is sufficient to consider the case when the initial deformation is uniform and the discontinuity surface is represented by the plane

$$\xi_1 = Vt, \quad V = \text{const}$$

We shall consider discontinuities of the shock wave type on which the displacement derivatives with respect to coordinates and time become discontinuous, together with the components of the strain and stress tensors, while the displacements themselves remain continuous. Then, using the continuity of $\mathbf{w}$ on the discontinuity surface, we obtain the so-called kinematic relations at discontinuities /1/

$$[\partial\mathbf{w}/\partial\xi_2] = 0, \quad [\partial\mathbf{w}/\partial\xi_3] = 0, \quad [\partial\mathbf{w}/\partial t] + V[\partial\mathbf{w}/\partial\xi_1] = 0, \qquad ([a] = a^+ - a^-)$$

where a^+ and a^- the values of a behind and ahead of the discontinuity. Since $\partial\mathbf{w}/\partial t = \mathbf{v}$ is the medium velocity, the last equation can be written as

$$[\mathbf{v}] = -V[\partial\mathbf{w}/\partial\xi_1] \tag{1.2}$$

The second group of conditions on the discontinuity represents the conservation of energy and momentum. With the use of the law of mass conservation we write them as /1/

$$[\sigma_{i1}] + \rho_0 V[v_i] = 0, \quad V[\Phi] + {}^1/_2\,\rho_0 V[v^2] + [v_i\sigma_{i1}] = 0 \tag{1.3}$$

Using formula (1.2) and expression (1.1) for σ_{ij}, we eliminate in the last equations for v_i and σ_{i1}, assuming that $v_i^- = 0$, and obtain

$$[\partial\Phi/\partial u_k] = \rho_0 V^2[u_k] \tag{1.4}$$

$$[\Phi] = {}^1/_2\,[\partial\Phi/\partial u_k][u_k] + (\partial\Phi/\partial u_k)^+[u_k] \tag{1.5}$$

where $u_k = \partial w_k/\partial\xi_1$ are components of tensor $\partial w_i/\partial\xi_j$ which become discontinuous. For specified state ahead of discontinuity and discontinuity velocity V, system (1.4), (1.5) enables us to determine the changes of u_k and of entropy S. These quantities can then be used together with equalities (1.2) and (1.1) for determining the velocities and stresses behind the shock. The totality of these quantities for the state behind the discontinuity corresponding to a specified state ahead of the latter constitute a set that depends on a single, constantly varying parameter for which we may take the quantity V. This set which in the space of quantities determined behind the discontinuity represents a curve or several segments of curves, is called here the Hugoniot curve.

2. Conditions on weak shock waves and conditions of entropy increase.

It was shown in /7/ that the variation of quantities in weak shock waves coincide within third order terms with respect to shock amplitude with the variation of quantities in respective simple wave. The entropy variation is, thus, a quantity of at least the third order of smallness relative to shock magnitude (this will be proved by direct calculations). Hence in the case of weak shock waves, which are investigated here, only the first order dependence of Φ on $[S]$ need to be taken into account and terms of the form $[u_k]^\alpha[S]$, $\alpha > 0$ may be disregarded. It is consequently possible to assume that $\Phi^+ - \Phi^-$ contains $[S]$ only in the form of the term $\rho_0 T_0[S]$. It is, then, possible to introduce the function

$$\Psi = \Phi^+ - \Phi^- - (\partial\Phi/\partial u_k)^-[u_k] - \rho_0 T_0[S] \tag{2.1}$$

which in the considered approximation is independent of $[S]$, and whose expansion in series in $[u_k]$ begins with quadratic terms. Equations (1.4) now assume the form

$$(\partial\Psi/\partial u_k)^+ = \rho_0 V^2[u_k] \tag{2.2}$$

These equations link the quantities $[u_k]$ and V. After some calculations we obtain from Eq. (1.5)

$$\rho_0 T_0[S] = {}^1/_2\,(\partial\Psi/\partial u_i)^+[u_i] - \Psi^+ \tag{2.3}$$

The condition of entropy growth at the discontinuity requires that the right-hand side of the last formula be positive.

Note the following corollary of Eqs. (2.2) and (2.3): the maxima of V on the Hugoniot curve coincide with the maxima of S, and this also holds for the minima (of these quantities).

We use the notation $\rho_0 V^2 = \alpha$, $[u_1] = x$, $[u_2] = y$ and $[u_3] = z$, and $\mathbf{r}$ for the vector with components x, y, and z. Equations (2.2) can now be integrated

$$\Psi^+ = \int_0^r \alpha(\xi)\xi\, d\xi$$

where ξ is the running value of $r = |\mathbf{r}|$ and α is assumed to be some function of r on the Hugoniot curve. Using the last formula and equality (2.2) we rewrite (2.3) in the form

$$\rho_0 T_0 (S - S_0) = {}^1/_2\, \alpha(r) r^2 - \int_0^r \alpha(\xi)\xi\, d\xi$$

whose differentiation yields

$$\rho_0 T_0 S'(r) = {}^1/_2\, \alpha'(r) r^2, \quad \rho_0 T_0 S''(r) = {}^1/_2\, \alpha''(r) r^2 + r\alpha'(r)$$

The first of these equalities shows that the first derivatives of S and V vanish simultaneously, and the second that the signs of second derivatives are the same. The first of these statements was proved in /8/ in a more general case.

3. Weak shock waves in an isotropic lightly deformed body. The linear approximation. Since the investigation is restricted to weak shock waves and small deformations, and even smaller entropy variations, we represent the elastic potential of an isotropic elastic body in the form of expansion

$$\Phi = {}^1/_2\, \lambda I_1^2 + \mu I_2 + l I_1 I_2 + m I_3 + n I_1^3 + \xi I_2^2 + \eta I_1 I_3 + \zeta I_1^2 I_2 + \varkappa I_1^4 + \rho_0 T_0 (S - S_0) + \text{const} \tag{3.1}$$

$$I_1 = \varepsilon_{ii}, \quad I_2 = \varepsilon_{ik}\varepsilon_{ik}, \quad I_3 = \varepsilon_{ik}\varepsilon_{kj}\varepsilon_{ji}$$

all of those coefficients are assumed constant. Formula (3.1) contains terms of up to fourth order with respect to ε_{ij}. Its application in the study of shock waves requires not only small variation of quantities in the wave but, also, small initial deformations. The last constraint may be disregarded but, as will be subsequently made clear, even in the case of small initial deformations some very interesting properties of transverse shock waves are disclosed /with it/. It is also possible to investigate shock waves in media for which the elastic potential formula does not contain ε_{ij} in powers higher than the fourth.

When the discontinuity intensity is fairly small, it can be determined in linear approximation by restricting the expression for Ψ obtained from (3.1) to quadratic terms

$$\Psi = dx^2 + fy^2 + gz^2 + kxy + hxz \tag{3.2}$$

where the coordinates ξ_2 and ξ_3 are selected so that the expression for Ψ does not contain the term yz. The coefficients in formula (3.2) are defined as follows:

$$d = {}^1/_2\lambda + \mu + 3aI_1 - (\lambda + 3\mu + 3m + 2l)(\varepsilon_{22}^\circ + \varepsilon_{33}^\circ), \quad f = {}^1/_2\,\mu + bI_1^\circ - (\mu + {}^3/_4\, m)\,\varepsilon_{33}^\circ, \quad k = 4b\varepsilon_{12}^\circ \tag{3.3}$$

$$g = {}^1/_2\,\mu + bI^\circ_1 - (\mu + {}^3/_4\, m)\,\varepsilon_{22}^\circ, \quad h = 4b\varepsilon_{13}^\circ, \quad a = {}^1/_2\,\lambda + \mu + l + m + n, \quad b = {}^1/_2\,\lambda + \mu + {}^1/_2\, l + {}^3/_4\, m$$

$$\varepsilon_{ij}^\circ = {}^1/_2\,(\partial w_i / \partial \xi_j + \partial w_j / \partial \xi_i), \quad I_1^\circ = \varepsilon_{ii}^\circ$$

In this case Eqs. (2.2) are of the form used for the determination of eigenvectors and eigenvalues of the matrix of coefficients of the quadratic form for Ψ. The eigenvectors define the variation of quantities in linear waves, while the eigenvalues determine the quantities $\alpha_i = \rho_0 V_i^2$ for such waves, i.e. their propagation velocities V_i, $i = 1, 2, 3$ which in linear approximation are the same as the characteristic velocities $c_i = d\xi_1 / dt$ in equations of the elasticity theory.

If the effect of initial deformations is disregarded in formulas (3.1), one eigenvector that corresponds to $\alpha = \rho_0 c_1^2 = \lambda + 2\mu$ lies on the x-axis and defines the variation of quantities in the longitudinal wave, and the remaining eigenvectors which correspond to the double eigenvalue $\alpha = \rho_0 c_2^2 = \mu$ fill the yz-plane and determine the variation of quantities in transverse waves.

In the considered approximation with allowance for initial deformations the surface $\Psi = \text{const}$ is generally a triaxial ellipsoid whose three principal axes define three mutually perpendicular characteristic dimensions that correspond to three different eigenvalues of α and, consequently, to the three different characteristic velocities c_i. Owing to the smallness of initial deformations, one of the characteristic directions lies close to the x-axis, while the remaining two are in a plane which is close to the yz-plane. In this approximation a jump from the initial point $x = 0$, $y = 0$, $z = 0$ to any point of one of the principal axes of the ellipsoid $\Psi = \text{const}$ is possible. In the considered linear approximation the set of three principal axes of the ellipsoid represent the Hugoniot curve.

When considering discontinuities in nonlinear approximation it is necessary to take into account in the expression for Ψ terms of higher order to make possible a more exact determination of the Hugoniot curve segments which issue from the coordinate origin. The tangents to these are the characteristic directions in the linear approximation considered above.

4. Quasilongitudinal waves. The nonlinear approximation. When considering Ψ with an accuracy to third power terms with respect to initial deformations and those acquired in the discontinuity, we have

$$\Psi = ax^3 + bx(y^2 + z^2) + dx^2 + fy^2 + gz^2 + kxy + hxz \tag{4.1}$$

where the coefficients are determined equalities (3.3). Assuming that $y \ll x$ and $z \ll x$, we can represent Eqs. (2.2) in the form

$$3ax^2 + 2dx = \alpha x, \quad 2bxy + 2fy + kx = \alpha y, \quad 2bxz + 2gz + hx = \alpha z$$

the first of which yields

$$\alpha \equiv \rho_0 V^2 = 2d + 3ax \tag{4.2}$$

The other two equations yield, after the elimination of α, the apertaining branch of the Hugoniot curve

$$y = \frac{kx}{2(d-f)}\left(1 - \frac{3a-2b}{2(d-f)}x\right), \quad z = \frac{hx}{2(d-g)}\left(1 - \frac{3a-2b}{2(d-g)}x\right)$$

These formulas show that quasilongitudinal waves always contain transverse shock components $(y \neq 0, z \neq 0)$ which, owing to the smallness of the initial deformation (k and h), are one order lower than those of the longitudinal wave. The entropy change calculated by formula (2.3) with function Ψ specified by equality (4.1), is defined as follows:

$$2\rho_0 T_0[S] = x(ax^2 + by^2 + bz^2) \tag{4.3}$$

Thus the condition of entropy increase in the shock in the case of a quasilongitudinal wave is of the form

$$ax \geqslant 0 \tag{4.4}$$

If the material is such that $a = {}^1/_2\lambda + \mu + l + m + n > 0$, only shocks with $x > 0$, i.e. rarefaction shocks, are possible in it, if however $a < 0$, only compression shocks are possible.

When $ax > 0$, the shock velocity increases with the increase of its intensity. In linear approximation (as $x \to 0$). The shock velocity becomes equal to the characteristic velocity $c_1 = \sqrt{2d/\rho_0}$. The shock intensity x can be used for determining all parameters behind the wave, including the characteristic velocity behind the discontinuity

$$c_1^+ = \sqrt{(2d + 6ax)/\rho_0} \tag{4.5}$$

5. Quasitransverse wave. Equality (4.3) shows that in a nearly transverse wave, when $x \ll \max(y, z)$, the entropy in the third approximation with respect to shock amplitude is constant. Determination of the entropy change thus necessitates, as in the case of absence of initial deformation /1/, an extension of all expansions to fourth order with respect to shock intensity. Because of this we supplement the function Ψ by fourth order terms with respect to y and z

$$\Psi = \delta(y^2 + z^2)^2 + (y^2 + z^2)(bx + py + qz) + dx^2 + fy^2 + gz^2 + kxy + hxz \tag{5.1}$$

$$\delta = {}^1/_4({}^1/_2\lambda + \mu + l + {}^3/_2 m + \xi), \quad p = 8\delta\varepsilon_{12}^\circ, \quad q = 8\delta\varepsilon_{13}^\circ$$

Eliminating α from Eq. (2.2) on the shock, we obtain the Hugoniot curve equation

$$x = -\omega[b(y^2 + z^2) + ky + hz] \tag{5.2}$$

$$A_0(kz - hy)(y^2 + z^2) + b\omega(kz - hy)(ky + hz) - 2b(f - g)yz = 0 \tag{5.3}$$

$$\left(\omega = \frac{1}{\lambda + \mu}, \quad A_0 = {}^1/_4\mu + \frac{1}{\lambda+\mu}({}^1/_2\mu + {}^1/_2 l + {}^3/_4 m)^2 - {}^1/_2\xi\right)$$

If even one of the quantities $k, h, f - g$ (i.e. $\varepsilon_{12}^\circ, \varepsilon_{13}^\circ, \varepsilon_{33}^\circ - \varepsilon_{22}^\circ$) are nonzero, Eq. (5.3) defines a curve which is the projection of the Hugoniot curve on the y, z-plane. We are basically concerned with this case. When all of these quantities are zero, shock transitions are possible at a point with arbitrary y and z /1/.

In polar coordinates $y = r\cos\theta$, $z = r\sin\theta$ Eq. (5.3) assumes the form

$$r = \frac{kh\omega b}{A_0}\frac{1}{\sin 2\theta_0}\frac{\sin 2(\theta - \theta_0)}{h\cos\theta - k\sin\theta} \tag{5.4}$$

where the angle θ_0 is determined by the equation

$$\operatorname{ctg} 2\theta_0 = n \equiv \frac{{}^1/_2(k^2 - h^2) + (g - f)\omega^{-1}}{kh} \tag{5.5}$$

accurate within terms that are multiples of $\pi/2$.

We number axes ξ_2 and ξ_3 so as to have the numerator in the right-hand side of (5.5) always nonnegative. Then the sign of n is the same as that of the quantity $k/h = \varepsilon_{12}^{\circ}/\varepsilon_{13}^{\circ}$.

The curve of Eq. (5.3) or (5.4) is shown in Fig.1. It is of the form of a loop with branches that approach infinity along the asymptote

$$z = \frac{h}{k} y + \frac{2(f-g)bh}{A_0(k^2+h^2)}$$

The curve has a crunode at point 0 where its branches are tangent to a new orthogonal axes y' and z' turned by angle θ_0 relative to the original axes. One of the curve branches must necessarily intersect the asymptote, while the other does not. Equation (5.4) implies that any straight line passing in the yz-plane through the coordinate origin intersects curve (5.3) only once.

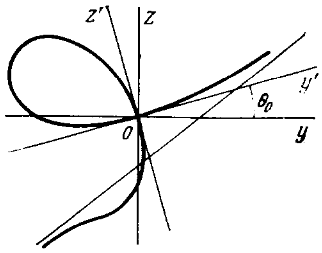

Fig.1

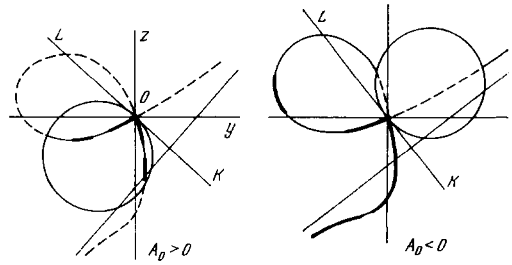

Fig.2

The maximum size of the loop depends on parameter n. Note that the coefficients in Eq. (5.3) are of different orders of smallness. Thus, A_0, ω, and b are, respectively, of orders of $1, k$ and h which is of the order of ε (initial strain), and $g - f = \left(\mu + \frac{3}{4} m\right)(\varepsilon_{33}^{\circ} - \varepsilon_{22}^{\circ}) + O(\varepsilon^2)$.

If the quantity $g - f$ is of the order of ε, the loop dimensions in the yz-plane are finite, and the considered expansions hold only for small y and z. Because of this, it is only possible to talk in this case of small sections of the curve near the coordinate origin. If the quantity $g - f$ is smaller than ε, the loop dimensions are small, and the loop is entirely contained in the region of theexpansion validity. This case is particularly interesting and is mainly considered here.

When $f - g = 0$ or $k = 0$, or $h = 0$, we obtain degenerate forms of curve (5.3) represented by a circle intersected by a straight line.

Let us now indicate those sections of the Hugoniot curve which correspond to an entropy increase. Using formulas (2.3) and (5.1), for quasitransverse waves, we obtain

$$2\rho_0 T_0 [S] = -A_0(y^2+z^2)\left(y^2+z^2+\frac{k}{b}y+\frac{h}{b}z\right) \geqslant 0$$

(the quantity A_0 has been previously defined). For materials whose elastic properties satisfy the condition $A_0 > 0$, the points lying within the circle

$$\left(y+\frac{k}{2b}\right)^2 + \left(z+\frac{h}{2b}\right)^2 = \frac{k^2+h^2}{4b} \tag{5.6}$$

correspond to entropy growth, while in the case of material with $A_0 < 0$, the region outside that circle corresponds to such growth.

Circle (5.6) passes through the coordinate origin and is tangent to line LK normal to the asymptote (Fig.2). The circle intersects the Hugoniot curve at point O and at two other points, one of which lie on the "loop" and the other on the "branch". The cases of $A_0 > 0$ and $A_0 < 0$ correspond to entropy circules lying on the opposite sides of the LK line. The curves shown in Fig.2 relate to the case when $n > 0$, $kh > 0$ and $f - g > 0$, and the dash lines represent there sections of the Hugoniot curve where $[S] < 0$. For other relations between parameters we obtain the same pattern but turned relative to initial axes by angles that are multiples of $\pi/2$, or as mirror reflections (when $n < 0$). Note that the radius of circle (5.6) is always of the order of ε, including the case when the loop is of finite dimensions.

6. Velocity of quasitransverse shock waves.

Using system (2.2) and formula (5.1) we can determine at every point of the Hugoniot curve the quasitransverse shock wave velocity

$$\alpha = \rho_0 V^2 = 2f\cos^2\theta + 2g\sin^2\theta - \omega(k\cos\theta + h\sin\theta)^2 - 3A_0 b^{-1} r(k\cos\theta + h\sin\theta) - 2A_0 r^2 \tag{6.1}$$

where r is taken from Eq. (5.4).

In the y'- and z'-axes drawn tangent to the loop at point 0, the expression for wave velocity is of the form

$$\alpha = f + g - \omega \frac{k^2 + h^2}{2} - \frac{\omega k h}{\sin 2\theta_0} + \frac{2\omega k h}{\sin 2\theta_0} \times \tag{6.2}$$

$$t\left[t(Nt - M)^2 + 3(Mt + N)(Nt - M) - 4t\frac{b^2\omega h k}{A_0 \sin 2\theta_0}\right] \times (1 + t^2)^{-1}(Nt - M)^{-2}$$

$$t = \operatorname{tg}\varphi,\ \ \varphi = \theta - \theta_0,\ \ M = h\cos\theta_0 - k\sin\theta_0,\ \ N = h\sin\theta_0 + k\cos\theta_0$$

In conformity with the definition of φ radius r approaches zero along the Hugoniot curve from two directions: $\varphi = 0$ and $\varphi = \pi/2$, which corresponds to two lineartransverse waves propagating at the characteristic velocities

$$a_{2,3} = \rho_0 (c_{2,3}^-)^2 = f + g - \omega \frac{k^2 + h^2}{2} \pm \frac{\omega k h}{\sin 2\theta_0} \tag{6.3}$$

As implied by (5.6), the sign of kh is always the same as that of $2\sin\theta_0$, so that when $kh \neq 0$ always $c_2^- > c_3^-$. It can be shown that higher characteristic velocities always correspond to that section of the Hugoniot curve branch where $[S] > 0$, and the lowest to the similar section of the loop. The remainder $c_2^- - c_3^- = 2\omega k h\sqrt{1 + n^2}$ is small, but the characteristic velocities c_2^- and c_3^- are not the same when $\varepsilon_{12}^{\circ}\varepsilon_{13}^{\circ} \neq 0$. The curve $r(\varphi)$ approaches infinity along the asymptote as $\varphi \to \varphi_* = \operatorname{arc\,tg}(M/N)$.

The shock wave velocity increases with increasing intensity.

7. Evolution of shock waves.

Entropy growth is not a sufficient condition for the possibility of shock wave occurrence. It is necessary to ascertain in addition that the conditions of correctness (or of evolution) of boundary conditions at the discontinuity /7,9, 10/ are satisfied. The conditions of evolution specify that the number of characteristics issuing from the discontinuity in both directions must be by one smaller than the number of boundary conditions at the discontinuity.

Since the velocity of a quasilongitudinal wave exceeds by a finite quantity the characteristic velocities of transverse waves, it is sufficient for the evolution of a weak quasilongitudinal wave that the conditions

$$c_1^- \leqslant V \leqslant c_1^+ \tag{7.1}$$

are satisfied. It follows from (4.2) and (4.5) that these conditions are satisfied when $ax \geqslant 0$. As previously shown, the last of these conditions ensures that the entropy in shock waves does not decrease. Thus in the approximation considered here, the condition of entropy growth and the conditions of evolution coincide in the case of longitudinal shock waves.

The conditions of evolution of quasitransverse waves stipulates that one of the following two systems of inequalities must be satisfied:

$$\begin{cases} c_2^- \leqslant V \leqslant c_2^+ \\ c_3^+ \leqslant V \leqslant c_1^- \end{cases}, \qquad \begin{cases} c_3^- \leqslant V \leqslant c_3^+ \\ 0 \leqslant V \leqslant c_2^- \end{cases} \tag{7.2}$$

where c_i^- and c_i^+ are characteristic velocities ahead and behind the shock wave, respectively. The numeration is selected that $c_3^- \leqslant c_2^- \leqslant c_1^-$ and $c_3^+ \leqslant c_2^+ \leqslant c_1^+$. The longitudinal wave characteristic velocity c_1^- exceeds c_2^- by a finite quantity, hence the inequality $V < c_1^-$ as well as the condition that $V > 0$ are always satisfied for a weak transverse wave.

Let us indicate the regions of validity of inequalities (7.2) on the Hugoniot curve. Using equality (6.1) we represent the dependence of the discontinuity velocity $V = \sqrt{\alpha/\rho_0}$ on the polar angle φ (Fig.3) in which the left diagram corresponds to $A_0 > 0$ and the right one to $A_0 < 0$.

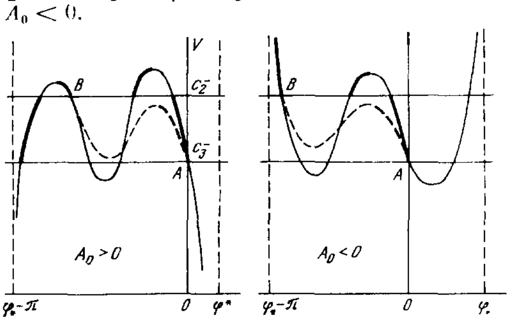

Fig.3

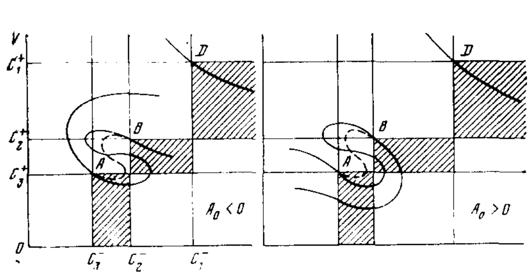

Fig.4

As previously indicated, the dependence of all quantities on φ is periodic of period π. One of such periods is shown in the diagrams; it ends with r approaching infinity along the Hugoniot curve branches, shown in Figs.1 and 2.

The segment of curve $V(\varphi)$ between points A and B corresponds to the loop in Figs.1 and 2, while its remaining part corresponds to branches. The points of intersection of that part of curve $V(\varphi)$, which corresponds to the loop, with the straight lines $V = c_2^-$ and $V = c_3^-$ shown in Fig.3 can be absent, depending on the properties of the medium and on the type of initial deformation. This case is presented in the diagrams by dashed lines. Using formula (6.2) it is possible to show that curve $V(\varphi)$ has always three extrema. As shown in /8/, points of the Hugoniot curve at which extremal velocities obtain are "Jouguet" points, where the wave velocity V is equal to one of the characteristic velocities c_i^+ behind the discontinuity and the remainder $V - c_i^+$ changes its sign at that point. Thus, the points of extremal values of V may be taken togehter with points at which the equalities $V = c_i^-$, as the boundaries of evolution regions. The parts of curves $V(\varphi)$ along which the evolution conditions (7.2) are satisfied are shown in Fig.3 by heavy lines. The respective parts of the Hugoniot curve are drawn in Fig.2 by solid lines (they correspond to the solid line curve in Fig.3).

As shown above, the maxima and minima of V coincide with the maxima and minima of S on the Hugoniot curve, which enables us to state that the evolutionary parts of curve $V(\varphi)$ adjacent to points A and B, and the respective parts of the Hugoniot curve always lie in the region $[S] > 0$. Indeed, as the distance from points A and B increases along that curve, the quantity $[S]$ increases.

The conditions of entropy growth must be checked along the remaining parts of the Hugoniot curve where the conditions of shock evolution are satisfied. On the other hand, not all parts of the Hugoniot curve where $[S] > 0$ automatically satisfy the conditions of evolution, since nonevolution sections of that curve always adjoint points at which $[S]$ reaches its maximum.

Relations between the shock-wave velocity and the characteristic velocities are conveniently represented curves, as in Fig.4 with c_i^- and c_i^+ are taken along the coordinate axes. The shaded rectangles correspond to regions where the evolution inequalities (7.2) are satisfied.

Points A, B and D at which $c_i^- = c_i^+$ correspond to initial states for shock waves. Points of the curve in Fig.4 represents the wave velocity in relation to characteristic velocities. At its intersections with vertical lines $V = c_i^-$ and at those with horizontal lines $V = c_i^+$. Sections of the curve that lie in shaded areas are evolutionary.

Note that when initial deformations are absent and $A_0 < 0$, only one part of the Hugoniot curve remains evolutionary for quasitransverse waves. In Fig.2 that part lies to the left of point B which in this case merges with point A. When $A_0 > 0$ and initial deformations are absent, the Hugoniot curve has no sections for the evolution of quasitransverse waves. Thus in the absence of initial deformations of evolution and of entropy growth coincide, and the evolution condition is generally the stronger one.

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